

# Zeno dynamics yields ordinary constraints

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(February 1, 2008)

The dynamics of a quantum system undergoing frequent measurements (quantum Zeno effect) is investigated. Using asymptotic analysis, the system is found to evolve unitarily in a proper subspace of the total Hilbert space. For spatial projections, the generator of the “Zeno dynamics” is the Hamiltonian with Dirichlet boundary conditions.

PACS numbers: 03.65.Bz; 03.65.Db; 02.30.Mv

Frequent measurement can slow the time evolution of a quantum system, hindering transitions to states different from the initial one [1,2]. This phenomenon, known as the quantum Zeno effect (QZE), follows from general features of the Schrödinger equation that yield quadratic behavior of the survival probability at short times [3].

However, a series of measurements does not necessarily freeze everything. On the contrary, for a projection onto a multi-dimensional subspace, the system can evolve away from its initial state, although it remains in the subspace defined by the “measurement.” This continuing time evolution *within* the projected subspace we call *quantum Zeno dynamics*. It is often overlooked, although it is readily understandable in terms of a theorem on the QZE [2] that we will recall below.

The aim of this article is to show that Zeno dynamics yields ordinary constraints. Under general conditions, the evolution of a system undergoing frequent measurements takes place in a proper subspace of the total Hilbert space and the wave function satisfies Dirichlet boundary conditions on the domain defined by the measurement process. Moreover, the evolution is generated by a self-adjoint Hamiltonian and remains reversible within the Zeno subspace. This shows that the irreversibility is not compulsory, as noted in [4].

The QZE has been tested on oscillating systems [5] and there has been a recent observation of non-exponential decay (leakage through a confining potential) at short times [6]. Although these experiments have invigorated studies on this issue, they deal with one-dimensional projectors (and therefore one-dimensional Zeno subspaces): the system is forced to remain in its initial state. This is also true for interesting quantum optical applications [7]. The present work therefore enters an experimentally uncharted area, although the property of being a multidimensional measurement is not at all exotic, and in particular applies to the most basic quantum measurement: position. The latter is the paradigm for the present work.

We introduce notation. Consider a quantum system,  $Q$ , whose states belong to the Hilbert space  $\mathcal{H}$  and whose evolution is described by the unitary operator  $U(t) = \exp(-iHt)$ , where  $H$  is a time-independent semi-bounded Hamiltonian. Let  $E$  be a projection operator that does not commute with the Hamiltonian,  $[E, H] \neq 0$ , and  $E\mathcal{H} = \mathcal{H}_E$  the subspace defined by it. The initial density matrix  $\rho_0$  of system  $Q$  is taken to belong to  $\mathcal{H}_E$ :

$$\rho_0 = E\rho_0 E, \quad \text{Tr } \rho_0 = 1. \quad (1)$$

The state of  $Q$  after a series of  $E$ -observations at times  $t_j = jT/N$  ( $j = 1, \dots, N$ ) is

$$\rho^{(N)}(T) = V_N(T)\rho_0 V_N^\dagger(T), \quad V_N(T) \equiv [EU(T/N)E]^N \quad (2)$$

and the probability to find the system in  $\mathcal{H}_E$  (“survival probability”) is

$$P^{(N)}(T) = \text{Tr} \left[ V_N(T)\rho_0 V_N^\dagger(T) \right]. \quad (3)$$

Our attention is focused on the limiting operator

$$\mathcal{V}(T) \equiv \lim_{N \rightarrow \infty} V_N(T). \quad (4)$$

Misra and Sudarshan [2] proved that if the limit exists, then the operators  $\mathcal{V}(T)$  form a one-parameter semigroup, and the final state is

$$\rho(T) = \lim_{N \rightarrow \infty} \rho_N(T) = \mathcal{V}(T)\rho_0 \mathcal{V}^\dagger(T). \quad (5)$$

The probability to find the system in  $\mathcal{H}_E$  is

$$\mathcal{P}(T) \equiv \lim_{N \rightarrow \infty} P^{(N)}(T) = 1. \quad (6)$$

This is the QZE. If the particle is constantly checked for whether it has remained in  $\mathcal{H}_E$ , it never makes a transition to  $(\mathcal{H}_E)^\perp$ .

A few comments are in order. First, the final state  $\rho(T)$  depends on the characteristics of the model investigated and on the measurement performed (the specific forms of  $V_N$  and  $\mathcal{V}$  depend on  $E$ ). Moreover, the physical mechanism that ensures the conservation of probabilities within the relevant subspace hinges on the short time behavior of the survival probability: probability leaks out of the subspace  $\mathcal{H}_E$  like  $t^2$  for short times. Since the infinite- $N$  limit suppresses this loss, one can inquire under what circumstances  $\mathcal{V}(T)$  actually forms a group, yielding reversible dynamics within the Zeno subspace.

In this article we show that Zeno dynamics for a position measurement yields a particular kind of dynamics within the subspace defined by that measurement, namely unitary evolution with the restricted Hamiltonian, and with the domain of that (self adjoint) operator defined by Dirichlet boundary conditions. This elucidates the reversible features of the evolution for a wide class of physical models. As a spinoff, our proof provides a rigorous regularization of the example considered in [4].

We start with the simplest spatial projection.  $Q$  is a free particle of mass  $m$  on the real line, and the measurement is a determination of whether or not it is in the interval  $A = [0, L] \subset \mathbb{R}$ . The Hamiltonian and the corresponding evolution operator are

$$H = \frac{p^2}{2m}, \quad U(t) = \exp(-itH). \quad (7)$$

$H$  is a positive-definite self-adjoint operator on  $L^2(\mathbb{R})$  and  $U(t)$  is unitary. We study the evolution of the particle when it undergoes frequent measurements defined by the projector

$$E_A = \int dx \chi_A(x) |x\rangle\langle x|, \quad (8)$$

where  $\chi_A$  is the characteristic function

$$\chi_A(x) = \begin{cases} 1 & \text{for } x \in A = [0, L] \\ 0 & \text{otherwise} \end{cases}. \quad (9)$$

Thus  $E_A$  is the multiplication operator by the function  $\chi_A$ . We study the following process. We prepare a particle in a state with support in  $A$ , let it evolve under the action of its Hamiltonian, perform frequent  $E_A$  measurements during the time interval  $[0, T]$ , and study the evolution of the system *within* the subspace  $\mathcal{H}_{E_A} = E_A \mathcal{H}$ . We will show that the dynamics in  $\mathcal{H}_{E_A}$  is governed by the evolution operator

$$\mathcal{V}(T) = \exp(-iT H_Z) E_A, \quad (10)$$

$$\text{with} \quad H_Z \equiv \frac{p^2}{2m} + V_A(x), \quad V_A(x) = \begin{cases} 0 & \text{for } x \in A \\ +\infty & \text{otherwise} \end{cases}. \quad (11)$$

This is the operator obtained in the limit (4). In other words, the system behaves as if it were confined in  $A$  by rigid walls, inducing the wave function to vanish on the boundaries  $x = 0, L$  (Dirichlet boundary conditions).

We now prove our assertion. Let the particle be initially ( $t = 0$ ) in  $A$ . We recall the propagator in the position representation [8,9]

$$\begin{aligned} G(x, t; y) &\equiv \langle x | E_A U(t) E_A | y \rangle = \chi_A(x) \langle x | U(t) | y \rangle \chi_A(y) \\ &= \chi_A(x) \sqrt{\frac{m}{2\pi i \hbar}} \exp \left[ \frac{im(x-y)^2}{2\hbar t} \right] \chi_A(y), \end{aligned} \quad (12)$$

where  $t = T/N$  is the time when the first measurement is carried out and the particle found in  $E_A$ . To study the properties of  $G$  we choose a complete basis in  $L^2(A)$

$$u_n(x) = \langle x | u_n \rangle = \sqrt{\frac{2}{L}} \sin \left( \frac{n\pi x}{L} \right) \quad (n = 1, 2, \dots). \quad (13)$$

When these functions define the eigenbasis of  $H$ ,  $H$  is self adjoint and

$$H|u_n\rangle = E_n|u_n\rangle, \quad E_n = \frac{\hbar^2 n^2 \pi^2}{2mL^2}, \quad (14)$$

so that  $H$  has Dirichlet boundary conditions. The matrix elements of  $G$  are

$$G_{mn}(t) \equiv \langle u_m | E_A U(t) E_A | u_n \rangle = \int_0^L dx \int_0^L dy u_m(x) \sqrt{\frac{m}{2\pi i \hbar t}} \exp \left[ \frac{im(x-y)^2}{2\hbar t} \right] u_n(y). \quad (15)$$

Let  $r = x - y$ ,  $R = (x + y)/2$  and  $\lambda = m/2\hbar t$ , so that

$$G_{mn}(\lambda) = \sqrt{\frac{\lambda}{i\pi}} \int_0^L dR \int_{-r_0(R)}^{r_0(R)} dr u_m(R + r/2) u_n(R - r/2) \exp [i\lambda r^2]. \quad (16)$$

where  $r_0(R) = L - |L - 2R|$ . We now use the asymptotic expansion

$$g(\lambda) = \sqrt{\frac{\lambda}{i\pi}} \int_{-a}^a dx f(x) e^{i\lambda x^2} = g_{\text{stat}}(\lambda) + g_{\text{bound}}(\lambda), \quad (17)$$

where

$$g_{\text{stat}}(\lambda) = f(0) + \frac{i}{4\lambda} f''(0) + O(\lambda^{-2}), \quad (18)$$

$$g_{\text{bound}}(\lambda) = \frac{e^{i\lambda a^2}}{2ia\sqrt{i\pi\lambda}} [f(a) + f(-a)] + O(\lambda^{-3/2}) \quad (19)$$

are the contributions of the stationary point  $x = 0$  and of the boundary, respectively. By expanding the inner integral in (16) as in (17)–(19) one gets

$$\begin{aligned} & \sqrt{\frac{\lambda}{i\pi}} \int_{-r_0(R)}^{r_0(R)} dr u_m(R + r/2) u_n(R - r/2) \exp [i\lambda r^2] \\ &= u_m(R) u_n(R) + \frac{i}{4\lambda} \frac{d^2}{dr^2} [u_m(R + r/2) u_n(R - r/2)]_{r=0} + O(\lambda^{-3/2}). \end{aligned} \quad (20)$$

(Note that the contribution of the boundary vanishes identically.) Using this result, we integrate by parts and after a straightforward calculation obtain

$$\begin{aligned} G_{mn}(t) &= \int_0^L dR \left[ u_m(R) u_n(R) - \frac{it}{\hbar} u_m(R) \frac{-\hbar^2}{2m} \frac{d^2}{dR^2} u_n(R) \right] + O(t^{3/2}) \\ &= \langle u_m | u_n \rangle - \frac{it}{\hbar} \langle u_m | \frac{p^2}{2m} | u_n \rangle + O(t^{3/2}) \\ &= \delta_{mn} \left( 1 - \frac{it}{\hbar} E_n \right) + O(t^{3/2}). \end{aligned} \quad (21)$$

With this formula we can carry out the limit required in Eq. (4). At time  $T$ , in the energy representation, the propagator becomes

$$\begin{aligned} \mathcal{G}_{mn}(T) &= \langle m | \mathcal{V}(T) | n \rangle \\ &= \lim_{N \rightarrow \infty} \sum_{n_1 \dots n_{N-1}} G_{mn_1}(T/N) G_{n_1 n_2}(T/N) \dots G_{n_{N-1} n}(T/N) \\ &= \delta_{mn} e^{-iTE_n/\hbar}. \end{aligned} \quad (22)$$

This is precisely the propagator of a particle in a square well with Dirichlet boundary conditions and proves (10)–(11). Note that the  $t^{3/2}$  contribution in (21) drops out in the  $N \rightarrow \infty$  limit since it appears as  $N \times O(1/N^{3/2})$ . It is worth emphasizing that although this result has been proved using the basis (13), the information obtained is a property of the propagator, and therefore holds true in general. Our choice was a matter of convenience. With a different basis

and nonvanishing boundary conditions, the dominant contribution of order  $\lambda^{-1/2}$  in  $g_{\text{bound}}(\lambda)$  would have given a nondiagonal term in (20)–(22), showing that the chosen basis is not the right eigenbasis of  $H_Z$  (i.e., for the limiting object).

This result can be generalized to a wide class of systems. Let

$$H = \frac{p^2}{2m} + V, \quad U(t) = \exp(-itH), \quad (23)$$

where  $V$  is a regular potential. (It may be unbounded from below, for example  $V(x) = Fx$ , although within  $A$  the total Hamiltonian  $H$  should be lower bounded.) The measurement performed is again application of the projector (8) and we study the short-time propagator

$$G(x, t; y) = \chi_A(x) \sqrt{\frac{m}{2\pi i \hbar t}} \exp\left[\frac{im(x-y)^2}{2\hbar t}\right] \exp\left[-\frac{it(V(x) + V(y))}{2\hbar}\right] \chi_A(y). \quad (24)$$

The basis to be used for representing the propagator is again that of the Hamiltonian with Dirichlet boundary conditions in  $[0, L]$

$$H|u_n\rangle = \left(\frac{p^2}{2m} + V\right)|u_n\rangle = E_n|u_n\rangle, \quad u_n(x)|_{x=0,L} = 0. \quad (25)$$

As before ( $r = x - y$ ,  $R = (x + y)/2$ ,  $\lambda = m/2\hbar t$ )

$$G_{mn}(\lambda) = \sqrt{\frac{\lambda}{i\pi}} \int_0^L dR \int_{-r_0(R)}^{r_0(R)} dr \, u_n\left(R + \frac{r}{2}\right) e^{-itV(R+r/2)/2\hbar} u_n\left(R - \frac{r}{2}\right) e^{-itV(R-r/2)/2\hbar} e^{i\lambda r^2}. \quad (26)$$

Using the asymptotic expansion (17)–(19), a calculation identical to the previous one yields

$$\begin{aligned} G_{mn}(t) &= \int_0^L dR \left[ u_n(R)u_m(R) - \frac{it}{\hbar} u_n(R) \left( \frac{-\hbar^2}{2m} \frac{d^2}{dR^2} + V(R) \right) u_m(R) \right] + O(t^{3/2}) \\ &= \langle u_n | u_m \rangle - \frac{it}{\hbar} \langle u_n | \left( \frac{p^2}{2m} + V \right) | u_m \rangle + O(t^{3/2}) \\ &= \delta_{nm} \left( 1 - \frac{it}{\hbar} E_n \right) + O(t^{3/2}) \end{aligned} \quad (27)$$

and the limiting propagator at time  $T$  again reads

$$\mathcal{G}_{mn}(T) = \delta_{nm} e^{-iT E_n / \hbar}. \quad (28)$$

Again, the simplicity of the proof is due to the choice of the basis (25), satisfying Dirichlet boundary conditions.

We have also obtained an improvement with respect to earlier approaches to this problem. The aforementioned theorem by Misra and Sudarshan [2] requires that the Hamiltonian be lower bounded from the outset. However, we need only require that the Hamiltonian be lower bounded in the Zeno subspace. Despite the fact that for unbounded potentials (like  $V = Fx$ )  $H$  may not be lower bounded on the real line, the evolution in the Zeno subspace is governed by the Hamiltonian

$$H_Z = \frac{p^2}{2m} + V_A(x), \quad V_A(x) = \begin{cases} V(x) & \text{for } x \in A \\ +\infty & \text{otherwise} \end{cases} \quad (29)$$

that can be lower bounded in  $A$ , yielding a *bona fide* group for the evolution operators.

The above calculation and conclusions can readily be generalized to higher dimensions, so long as the measurement projects onto a set in  $\mathbb{R}^n$  with a smooth boundary (except at most a finite number of points). We again take  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$  and let the measurement-projection be defined by  $A \subset \mathbb{R}^n$ , which is not necessarily bounded. Again setting  $\mathbf{r} = \mathbf{x} - \mathbf{y}$ ,  $\mathbf{R} = (\mathbf{x} + \mathbf{y})/2$ , Eq. (26) becomes

$$G_{mn}(\lambda) = \left( \frac{\lambda}{i\pi} \right)^{n/2} \int_A d\mathbf{R} \int_{D(\mathbf{R})} d\mathbf{r} \, u_n(\mathbf{R} + \mathbf{r}/2) e^{-itV(\mathbf{R}+\mathbf{r}/2)/2\hbar} u_n(\mathbf{R} - \mathbf{r}/2) e^{-itV(\mathbf{R}-\mathbf{r}/2)/2\hbar} e^{i\lambda \mathbf{r}^2}, \quad (30)$$

where  $D(\mathbf{R})$  is the transformed integration domain for  $\mathbf{r}$ . The  $n$ -dimensional asymptotic expansions (17)–(19) read [10]

$$g_{\text{stat}}(\lambda) = f(\mathbf{0}) + \frac{i}{4\lambda} \Delta f(\mathbf{0}) + O(\lambda^{-2}), \quad (31)$$

$$g_{\text{bound}}(\lambda) = O(\lambda^{-1/2}) \times f(\text{boundary}) + O(\lambda^{-3/2}) \quad (32)$$

and the theorem follows again because  $f$  vanishes on the boundary (Dirichlet). The proof is readily generalized to non-convex and/or multiply-connected projection domains, the only difficulty being that the integration domain in (30) must be broken up. It is interesting to notice that at those points at which the boundary fails to have a continuously turning tangent plane, the asymptotic contribution of the discontinuity in the boundary in (32) would be of yet higher order in  $\lambda$ .

In conclusion, for traditional position measurements, namely projections onto spatial regions, we have shown that Zeno dynamics uniquely determines the boundary conditions, and that they turn out to be of Dirichlet type. This is also relevant for problems related to the consistent histories approach [11–13], where different boundary conditions were proposed. For us, the frequent imposition of a projection, the traditional idealization of a measurement, provides all the decohering of interfering alternatives that is needed. On the other hand, in the works just cited one seeks a restricted propagator (using the path decomposition expansion [14]) and such interference can occur. A second issue discussed in these works (especially [12]) is the validity of the Trotter product formula in certain cases. Again, our implicit use of this formula (in Eq. (12), etc.) is nothing more than its use for a free particle (or a particle in an ordinary potential). This is because the propagator of Eq. (12) provides time evolution under a *sequence* of operations: the particle evolves freely (on the entire line) for a time  $t$ , and *then* one applies the projection (left and right multiplication by the operator  $E_A$ ).

The present work has implications for the notion of “hard wall,” as used for example in elementary quantum mechanics. Everyone would agree (we expect) that this notion is an idealization. However, in many cases where this idealization is useful the “wall” is dynamic rather than static, the result of some fluctuating atomic presence. In this article we have a sufficient condition for the validity of this notion in a dynamic situation. Moreover, there is a quantitative framework (arising from our asymptotic analysis and finite-time-interval QZE effects) for gauging the effects of less than perfect hard walls.

## ACKNOWLEDGMENTS

This work is supported in part by the TMR-Network of the European Union “Perfect Crystal Neutron Optics” ERB-FMRX-CT96-0057 and by the United States NSF under grant PHY 97 21459.

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